

Exactness of supersymmetric WKB method for translational shape invariant potentials

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2003 J. Phys. A: Math. Gen. 36 5045

(<http://iopscience.iop.org/0305-4470/36/18/309>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.103

The article was downloaded on 02/06/2010 at 15:27

Please note that [terms and conditions apply](#).

Exactness of supersymmetric WKB method for translational shape invariant potentials

K M Cheng, P T Leung and C S Pang

Department of Physics, The Chinese University of Hong Kong, Shatin, Hong Kong SAR, People's Republic of China

E-mail: ptleung@phy.cuhk.edu.hk

Received 9 October 2002, in final form 13 February 2003

Published 23 April 2003

Online at stacks.iop.org/JPhysA/36/5045

Abstract

By examining the generic form of the superpotential of translational shape invariant potentials (TSIPs), we explicitly show the exactness of the lowest order supersymmetric WKB (SWKB) formula for TSIPs. Remarkably, our method applies to both unbroken and broken supersymmetric systems. We also demonstrate the equivalence of one-parameter and multi-parameter TSIPs, thus establishing the exactness of the SWKB formula for all TSIPs.

PACS numbers: 03.65.Ge, 03.65.Sq, 03.65.Fd

1. Introduction

In supersymmetric quantum mechanics (SUSY QM) [1–5], one considers a pair of Schrödinger equations (in units of $2m = 1$):

$$\hat{H}_1 \psi_n^{(1)} = \left[-\hbar^2 \frac{d^2}{dx^2} + V_1(x) \right] \psi_n^{(1)} = A^\dagger A \psi_n^{(1)} = E_n^{(1)} \psi_n^{(1)} \quad (1.1)$$

and

$$\hat{H}_2 \psi_n^{(2)} = \left[-\hbar^2 \frac{d^2}{dx^2} + V_2(x) \right] \psi_n^{(2)} = A A^\dagger \psi_n^{(2)} = E_n^{(2)} \psi_n^{(2)} \quad (1.2)$$

where $E_0^{(i)} < E_1^{(i)} < E_2^{(i)} < \dots$ ($i = 1, 2$), and the two operators A and A^\dagger are related to a superpotential W through the following equations:

$$A^\dagger = -\hbar \frac{d}{dx} + W(x) \quad A = \hbar \frac{d}{dx} + W(x). \quad (1.3)$$

It is obvious that the superpotential W has to satisfy the equations

$$V_1(x) = W^2(x) - \hbar W'(x) \quad (1.4)$$

and

$$V_2(x) = W^2(x) + \hbar W'(x). \quad (1.5)$$

Since

$$\hat{H}_1 (A^\dagger \psi_n^{(2)}) = A^\dagger A A^\dagger \psi_n^{(2)} = E_n^{(2)} (A^\dagger \psi_n^{(2)}) \quad (1.6)$$

$$\hat{H}_2 (A \psi_n^{(1)}) = A A^\dagger A \psi_n^{(1)} = E_n^{(1)} (A \psi_n^{(1)}) \quad (1.7)$$

these two supersymmetric partner Hamiltonians, \hat{H}_1 and \hat{H}_2 , are isospectral (or almost isospectral) and their normalized eigenfunctions can be related by the simple transformations:

$$\psi_{n+1}^{(1)}(x) = (E_{n+1}^{(1)})^{-1/2} A^\dagger \psi_n^{(2)}(x) \quad (1.8)$$

$$\psi_n^{(2)}(x) = (E_n^{(2)})^{-1/2} A \psi_{n+1}^{(1)}(x). \quad (1.9)$$

For unbroken supersymmetry where $E_0^{(1)} = 0$ and $A \psi_0^{(1)} = 0$, the eigenstate of \hat{H}_2 corresponding to $\psi_0^{(1)}$ is missing and the relation

$$E_{n+1}^{(1)} = E_n^{(2)} \quad (1.10)$$

holds for $n = 0, 1, 2, 3, \dots$. The two systems are then almost isospectral. In this case, the superpotential W is generated by the ground state wavefunction of \hat{H}_1 :

$$W(x) = -\hbar \frac{d}{dx} \ln \psi_0^{(1)}(x) = -\frac{\hbar [\psi_0^{(1)}]'}{\psi_0^{(1)}}. \quad (1.11)$$

Since the ground state wavefunction is nodeless, W so chosen is free of singularity.

Furthermore, there exists a special class of potentials known as shape invariant potentials (SIPs) [6], satisfying the relation

$$V_2(x; a_1) = V_1(x; a_2) + R(a_1) \quad (1.12)$$

where a_1, a_2 are free parameters characterizing the partner potentials V_1, V_2 , and $a_2 = f(a_1)$. Thus, V_1 and V_2 are similar in shape and R is called the remainder term that depends only on a_1 . One can then apply the SUSY transformation $n - 1$ times on V_1 to obtain a Hamiltonian \hat{H}_n which is almost isospectral with \hat{H}_1 except with the lowest $n - 1$ states missing. From the relation

$$V_n(x; a_1) = V_1(x; a_n) + \sum_{k=1}^{n-1} R(a_k) \quad (1.13)$$

it is then obvious that the ground state energy of $\hat{H}_n \equiv -\hbar^2 d^2/dx^2 + V_n$ is given by

$$E_0^{(n)} = \sum_{k=1}^{n-1} R(a_k). \quad (1.14)$$

By going back from \hat{H}_n to \hat{H}_1 one can show that the entire spectrum of \hat{H}_1 is given by

$$E_0^{(1)} = 0 \quad E_n^{(1)} = \sum_{k=1}^n R(a_k) \quad \text{for } n \geq 1. \quad (1.15)$$

SIPs play a very important role in SUSY QM because both the energies and wavefunctions of these systems can be exactly solved by the algebraic method. In fact, many exactly solvable potentials frequently encountered in mathematical physics belong to the class of SIPs [4].

Moreover, SIPs can be categorized according to the function $f(a_1)$. If $f(a_1) = a_1 + \alpha$, where α is an arbitrary constant, the family of SIPs is said to be related by translation, termed as translational SIPs (TSIPs) here. If, however, $f(a_1) = \alpha a_1$, they are said to be related by scaling [7, 8].

On the other hand, Comtet *et al* [9] have modified the usual WKB approximation [10] and obtained a supersymmetric WKB (SWKB) approximation that applies to SUSY partner potentials with unbroken symmetry. The quantization rule is given explicitly by

$$\int_{x_1}^{x_2} \sqrt{E_n^{(1)} - W^2(x)} dx = n\pi\hbar \quad \text{for } n = 0, 1, 2, \dots \quad (1.16)$$

where x_1, x_2 are the turning points defined by $E_n^{(1)} = W^2(x_1) = W^2(x_2)$. This SWKB formula can be obtained from the traditional WKB quantization rule [10]:

$$\int_{x'_1}^{x'_2} \sqrt{E_n^{(1)} - V_1(x)} dx = \left(n + \frac{1}{2}\right) \pi\hbar \quad (1.17)$$

where x'_1, x'_2 are the 'turning points' defined by $V(x'_1) = V(x'_2) = E_n^{(1)}$. The lowest order SWKB formula (1.16) then follows directly from an expansion of (1.17) in power series of \hbar [9].

Analogous to the conventional WKB method, the SWKB method gives accurate results for high lying excited states. Besides, the SWKB quantization rule is also exact for the ground state $n = 0$, yielding the correct value $E_0^{(1)} = 0$. Therefore, the SWKB method is expected to outperform the WKB method near the low energy regime. More remarkably, it was found later that the lowest order SWKB approximation readily gives the exact result for all SIPs known in the eighties, which are all of the translational type [9, 11]. The reason for the exactness of the SWKB formula is nontrivial and has been controversial since its discovery. Dutt *et al* first claimed in their paper [11] that the exactness originates from the level degeneracy between the partner systems that is preserved under the SWKB approximation. However, several years after the publication of Dutt *et al*'s paper, the lowest order SWKB formula was found to yield inexact results for SIPs related by scaling [12], which was discovered in the early nineties [7, 8]. Dutt *et al*'s argument, at least in its original form, failed to explain why the SWKB formula becomes inexact for such SIPs. It was soon recognized that the nontrivial dependence of the superpotential W on the Planck constant \hbar for these SIPs is the culprit [12]. After that, there were also some other attempts to study the exactness of the SWKB method [12–16].

The situation becomes even more surprising for potentials with broken supersymmetry (BSUSY) where neither the ground state energy of \hat{H}_1 nor that of \hat{H}_2 vanishes. The SWKB formula for BSUSY reads [17–20]:

$$\int_{x_1}^{x_2} \sqrt{E_n^{(1)} - W^2(x)} dx = \left(n + \frac{1}{2}\right) \pi\hbar \quad \text{for } n = 0, 1, 2, \dots \quad (1.18)$$

It is noteworthy that the above formula is in general inexact even for the ground state energy. However, this SWKB formula becomes exact for all states of TSIPs where BSUSY prevails [17–20]. As yet the connection between the exactness of the SWKB formula in SUSY and BSUSY systems has been shrouded in mystery.

The main objective of this paper is to study these controversial issues by examining the generic behaviour of the superpotential W . Adopting the results of factorization method, first proposed by Schrödinger and later by Infeld and Hull [21–24], we find that the superpotentials of all TSIPs can be unified and then succeed in showing explicitly the exactness of the SWKB formula for such potentials. In addition, the exactness of the SWKB formula is proved for both SUSY and BSUSY systems in a unified manner, thus providing a logical and intuitive link between equations (1.16) and (1.18).

The organization of our paper is as follows. In section 2 we reproduce the original proof proposed by Dutt *et al* [11] and discuss its validity. Section 3 describes a unified categorizing scheme for TSIPs for one-parameter TSIPs. We then establish explicitly the exactness of the SWKB formula for SUSY and BSUSY systems in sections 4 and 5, respectively. In section 6, we thoroughly examine the family of n -parameter TSIPs [24, 25] and show that they are in fact equivalent to one-parameter TSIPs. Consequently, our proof for the exactness of the SWKB formula for TSIPs is then completed and we end this paper with a brief discussion.

2. Conventional approach

The conventional proof [11] of the exactness of the lowest order SWKB formula is based on the common viewpoint that the SWKB quantization rule (a) preserves the level degeneracy between the partner potentials, i.e. $E_{n+1}^{(1)} = E_n^{(2)}$, and (b) yields the exact ground state energy, which is reproduced as follows. Consider the s th Hamiltonian of a family of SIPs, $H^{(s)}$, given explicitly by

$$H^{(s)} = -\hbar^2 \frac{d^2}{dx^2} + V_1(x, a_s) + \sum_{k=1}^{s-1} R(a_k). \quad (2.1)$$

From the SWKB formula for $H^{(s)}$:

$$\int \sqrt{E_n^{(s)} - W^2(x, a_s) - \sum_{k=1}^{s-1} R(a_k)} dx = n\pi\hbar \quad (2.2)$$

the ground state energy of $H^{(s)}$ is $E_0^{(s)} = \sum_{k=1}^{s-1} R(a_k)$, which is exact. If the SWKB really preserves level degeneracy between $H^{(s)}$ and $H^{(s-1)}$, which is a subtle point, then the energy of the first excited state of $H^{(s-1)}$ is $E_1^{(s-1)} = E_0^{(s)} = \sum_{k=1}^{s-1} R(a_k)$ and is also exact. One can then continue this procedure to the first Hamiltonian, $H^{(1)}$, and obtain exact energies for the first s states of $H^{(1)}$, namely, $E_n^{(1)} = \sum_{k=1}^n R(a_k)$ ($n = 1, 2, \dots, s$).

It is worthwhile to remark that in the foregoing derivation of the exactness of the SWKB formula the type of SIPs (related either by translation or by scaling) does not really matter. However, it was shown later that the SWKB formula is inexact for SIPs related by scaling [12]. This clearly unveils that some crucial point has been overlooked in the derivation sketched above.

We note here that the preservation of level degeneracy of the SWKB rule, in which the foregoing argument is rooted, is likely a flaw in the proof. In the application of the SWKB quantization rule to a pair of SUSY partner potentials, V_1 and V_2 , one uses two different formulae for each of them. (a) For $V_1 = W^2 - (\hbar/\sqrt{2m})W'$:

$$\int \sqrt{E_n^{(1)} - W^2(x, a_1)} dx = n\pi\hbar \quad n = 0, 1, 2, \dots \quad (2.3)$$

and (b) for $V_2 = W^2 + (\hbar/\sqrt{2m})W'$:

$$\int \sqrt{E_n^{(2)} - W^2(x, a_1)} dx = (n+1)\pi\hbar \quad n = 0, 1, 2, \dots \quad (2.4)$$

Despite the fact that these two formulae obviously preserve the level degeneracy of the two partner systems, namely $E_{n+1}^{(1)} = E_n^{(2)}$, they are not necessarily concomitant with (2.2), which is able to yield the exact ground state energy. Specifically, by letting $s = 2$ in (2.2), one obtains

$$\int \sqrt{E_n^{(2)} - W^2(x, a_2) - R(a_1)} dx = n\pi\hbar \quad n = 0, 1, 2, \dots \quad (2.5)$$

It is by no means obvious that (2.4) and (2.5) could produce identical results for an arbitrary SIP. Hence, the two conditions (a) preservation of level degeneracies and (b) exactness of ground state energy, from which the exactness of the SWKB formula follows [9], might not be consistent with each other. In fact, straightforward numerical computations readily show that they give different energies for SIPs related by scaling [12]. This strongly suggests that the conventional proof for the exactness of the SWKB formula is self-inconsistent.

3. Translational shape invariant potentials

We first consider TSIPs with one parameter, which are known exactly [23, 24]. Despite the fact that TSIPs with multiple parameters have been proposed previously [24, 25], the exact forms of such TSIPs have never been worked out to the best of our knowledge. In fact, we will show in the later part of our paper that TSIPs with multiple parameters are reducible to one-parameter TSIPs.

The shape invariant condition for SIPs related by translation is

$$V_2(x; a_s) = V_1(x; a_{s+1}) + R(a_s) \tag{3.1}$$

where $a_{s+1} = a_s + l$ with l being a constant. For convenience we also scale the parameter a_s so that $a_{s+1} = a_1 + s$, and let $\hbar = 1$ in the following derivation.

From the SUSY relation $V_{1,2} = W^2 \mp W'$, it is readily shown that [23]:

$$W^2(a_s, x) - W^2(a_{s+1}, x) + \frac{dW(a_s, x)}{dx} + \frac{dW(a_{s+1}, x)}{dx} = R(a_s). \tag{3.2}$$

Following the factorization method for TSIPs [23, 24], we expand the superpotential $W(x, a_s)$ in power series of a_s , whose expansion coefficients are functions of x . To illustrate the expansion, we firstly consider a trial solution linear in a_s :

$$W(a_s, x) = k_0(x) + a_s k_1(x). \tag{3.3}$$

Substitution of this form of W into (3.2) yields

$$R(a_s) = (2a_s + 1) (k'_1 - k_1^2) + 2(k'_0 - k_0 k_1). \tag{3.4}$$

Since $R(a_s)$ is independent of x , the coefficient of each term in the polynomial expansion of $R(a_s)$ must be a constant. That is,

$$k'_1 - k_1^2 = -\tilde{\alpha}^2 \tag{3.5}$$

and

$$k'_0 - k_0 k_1 = D. \tag{3.6}$$

Therefore, from equation (1.12) the eigenenergy $E_s^{(1)}(a_1)$ takes the form

$$E_s^{(1)}(a_1) = -\tilde{\alpha}^2[(s + a)^2 - a^2] \tag{3.7}$$

where

$$a = a_1 - \frac{D}{\tilde{\alpha}^2}. \tag{3.8}$$

It is then obvious that the constants $\tilde{\alpha}^2$ and D are real numbers because of the reality of $E_s^{(1)}(a_1)$. Since $E_s^{(1)} \geq 0$ increases monotonically with s , we conclude that (i) if $\tilde{\alpha}^2 > 0$, then $a < 0$ and there is a finite number of bound states; and (ii) if $\tilde{\alpha}^2 < 0$, then $a > 0$ and there is an infinite number of bound states. If both $\tilde{\alpha}$ and D equal zero, the solution is trivially $W = \text{constant}$, which corresponds to the free particle case and is ignored hereafter.

From (3.5) we have

$$\frac{dk_1}{k_1^2 - \tilde{\alpha}^2} = dx \quad (3.9)$$

whose general solution is

$$k_1(x) = -\tilde{\alpha} \frac{e^{\tilde{\alpha}x} + \eta e^{-\tilde{\alpha}x}}{e^{\tilde{\alpha}x} - \eta e^{-\tilde{\alpha}x}} \quad (3.10)$$

with η being an integration constant. To solve (3.6) in a systematic way, we substitute $k_0(x) = \tilde{k}_0(x)k_1$ into it and make use of (3.9) to obtain

$$\frac{d\tilde{k}_0}{D + \tilde{\alpha}^2 \tilde{k}_0} = \frac{dk_1}{k_1(k_1^2 - \tilde{\alpha}^2)}. \quad (3.11)$$

Direct integration of this equation yields a useful relation between the two functions k_1 and k_0 :

$$k_0(x) = b\sqrt{k_1^2 - \tilde{\alpha}^2} - \frac{D}{\tilde{\alpha}^2}k_1 \quad (3.12)$$

with b being another integration constant. Therefore, the superpotential can be neatly expressed in terms of the function k_1 , namely

$$W(x, a_{s+1}) = b\sqrt{k_1^2 - \tilde{\alpha}^2} + (a + s)k_1 \quad (3.13)$$

which actually includes the three classes of potentials considered in [6] as its particular solutions. We will make use of this form of the superpotential and also (3.9) to establish the exactness of the SWKB formula.

It is known that all the translational SIPs can be classified into six types (A, B, C, D, E and F) according to the forms of W and R [23]. While the general solution presented above is identified as the type A factorization, types B, C and D are all limiting cases of the former. For example, if $\eta = 1$ (other choices of positive η merely redefine the origin of the system) and $\tilde{\alpha} = \alpha > 0$, the TSIP generated is the generalized Pöschl–Teller potential with its potential, superpotential and energy being, respectively, given by

$$\begin{aligned} V_1(x, a_n) &= A_n^2 + (B^2 + A_n^2 + A_n\alpha) \operatorname{cosech}^2 \alpha x - (2A_n + \alpha)B \coth \alpha x \operatorname{cosech} \alpha x \\ W(x, a_n) &= A_n \coth \alpha x - B \operatorname{cosech} \alpha x \\ E_n(a_1) &= A_1^2 - A_n^2. \end{aligned} \quad (3.14)$$

Here $A_1 > 0$, $A_n = A_1 - (n - 1)\alpha$ and $B = -b\alpha \geq 0$ [26]. Likewise, if $\eta = -1$ while the other quantities remain unchanged, the Scarf II hyperbolic potential is obtained [26]:

$$\begin{aligned} V_1(x, a_n) &= A_n^2 + (B^2 - A_n^2 - A_n\alpha) \operatorname{sech}^2 \alpha x + (2A_n + \alpha)B \tanh \alpha x \operatorname{sech} \alpha x \\ W(x, a_n) &= A_n \tanh \alpha x + B \operatorname{sech} \alpha x \\ E_n(a_1) &= A_1^2 - A_n^2 \end{aligned} \quad (3.15)$$

where $B = b\alpha > 0$. On the other hand, if we replace $\tilde{\alpha}$ by an imaginary quantity $i\alpha$ in the above discussion, the Scarf I trigonometric potential will be obtained [26].

In addition to the general type A factorization, there are types B, C and D, which are limiting cases of type A. For example, in the type B case, $\tilde{\alpha} = \alpha > 0$ is a real number and η tends to zero and hence

$$\begin{aligned} \text{type B: } V_1(x, \alpha_1) &= A_n^2 + B_n^2 \exp(-2\alpha x) - 2B_n(A_n + \alpha/2) \exp(-\alpha x) \\ W(x, \alpha_n) &= A_n - B_n \exp(-\alpha x) \\ E_n(\alpha_1) &= A_1^2 - A_n^2 \end{aligned} \quad (3.16)$$

where $A_1, B \geq 0$, and $A_n = A_1 - (n - 1)\alpha$. This is simply the Morse potential [26].

Similarly, the type C and D factorizations result from the limit $\tilde{\alpha} \rightarrow 0$, respectively yielding the three-dimensional and one-dimensional shifted oscillators [23]. Therefore, B, C and D type factorizations can all be considered as appropriate limiting cases of the type A factorization and these four types of potentials can be treated as a single category.

On the other hand, similar results can be found if the negative power term in α is considered. If the trial solution is chosen as

$$W(x; a_s) = \frac{k_{-1}}{a_s} + k_0 + a_s k_1 \tag{3.17}$$

from (3.2) one can show that

$$\begin{aligned} R(a_s) = & (2a_s + 1)(k'_1 - k_1^2) + k_{-1}^2 \left(\frac{1}{a_s^2} - \frac{1}{(a_s + 1)^2} \right) - 2k_0 k_1 + 2k_0 \\ & + k'_{-1} \left(\frac{1}{a_s} + \frac{1}{a_s + 1} \right) + 2k_0 k_{-1} \left(\frac{1}{a_s} - \frac{1}{a_s + 1} \right). \end{aligned} \tag{3.18}$$

By the same argument that R is independent of x , each term with a different order in a_s in (3.18) must be equal to a constant. Multiplying this equation by $a_s^2(a_s + 1)^2$, we have

$$\begin{aligned} a_s^2(a_s + 1)^2 R(a_s) = & (2a_s^3 + a_s^2)(a_s + 1)^2(k'_1 - k_1^2) + (a_s^2 + \alpha_s)^2(k_0 - 2k_0 k_1) \\ & + (a_s^2 + \alpha_s)(2a_s + 1)k'_{-1} + (a_s^2 + \alpha_s)2k_0 k_1 - k_{-1}^2. \end{aligned} \tag{3.19}$$

Thus from the lowest order term we can conclude that

$$k_{-1} = b = \text{constant}. \tag{3.20}$$

Now from the term containing $k_0 k_{-1}$, we know that k_0 must be a constant. Furthermore, if k_0 is nonzero, the term containing $k_0 k_1$ also demands that k_1 is a constant and the solution is trivial. So we only consider the nontrivial case $k_0 = 0$, and the only remaining term is

$$k'_1 - k_1^2 = -\tilde{\alpha} = \text{constant} \tag{3.21}$$

which is identical to (3.5) and hence has a solution given by (3.10). The general solution of W is hence

$$W(x, a_s) = -a_s \tilde{\alpha} \frac{e^{\tilde{\alpha}x} + \eta e^{-\tilde{\alpha}x}}{e^{\tilde{\alpha}x} - \eta e^{-\tilde{\alpha}x}} + \frac{b}{a_s}. \tag{3.22}$$

If $\tilde{\alpha} \neq 0$, this is called type E factorization. If $\tilde{\alpha} = \alpha$, a positive real number, and $\eta = 1$, the superpotential takes the form

$$W(x, \alpha_n) = -A_n \coth(\alpha x) + \frac{B}{A} \tag{3.23}$$

where $A_n = a_n \alpha$ and $B = b \alpha$. Thus,

$$V_1(x, a_n) = A_n^2 - \frac{B^2}{A_n^2} - 2B \coth \alpha x + A_n(A_n - \alpha) \operatorname{cosech}^2 \alpha x \tag{3.24}$$

which is the Eckart potential [26], with an energy spectrum given by

$$E_n(a_1) = A_1^2 - (A_1 + n\alpha)^2 + B^2 \left[\frac{1}{A_1^2} - \frac{1}{(A_1 + n\alpha)^2} \right]. \tag{3.25}$$

If, instead, $\eta = -1$, it results in the Rosen–Morse hyperbolic potential II [26]:

$$V_1(x, a_n) = A_n^2 + \frac{B^2}{A_n^2} + 2B \tanh \alpha x - A_n(A_n + \alpha) \operatorname{sech}^2 \alpha x \tag{3.26}$$

with $A_n = -a_n\alpha$, and the energy spectrum is given by

$$E_n(a_1) = A_1^2 - (A_1 - n\alpha)^2 + B^2 \left[\frac{1}{A_1^2} - \frac{1}{(A_1 - n\alpha)^2} \right]. \quad (3.27)$$

It is worthwhile to note that the Rosen–Morse potential I can be obtained if $\tilde{\alpha}$ is equal to an imaginary number $i\alpha$. [26]. Lastly, type F factorization can be considered as the limiting case of $\tilde{\alpha} \rightarrow 0$, which results in the Coulomb potential.

We have considered in the above discussion two specific forms of the superpotential of TSIPs. However, it has been pointed out that other power series expansions in a_s for the superpotential are all incompatible with the asymptotic behaviour of E_n [24, 27]. Furthermore, if higher order terms are included in the trial solution of W , i.e.

$$W(a_s, x) = k_0(x) + a_s k_1(x) + a_s^2 k_2(x) + \cdots + a_s^n k_n(x) \quad (3.28)$$

for finite $n > 1$, equation (3.2) becomes

$$\begin{aligned} & [(a_s + 1)^{2n} k_n^2 + (a_s + 1)^{2n-1} (k_n k_{n-1}) + \cdots] + [(a_s + 1)^n k'_n + (a_s + 1)^{n-1} k'_{n-1} + \cdots] \\ & - [a_s^{2n} k_n^2 + a_s^{2n-1} (k_n k_{n-1}) + \cdots] + [a_s^n k'_n + a_s^{n-1} k'_{n-1} + \cdots] = \text{constant}. \end{aligned} \quad (3.29)$$

From the leading term of a_s , we know that k_n is a constant. Similarly, from the next order term of a_s , k_{n-1} is also shown to be a constant. If we continue the procedure it is not difficult to show that all k_r for $r > 0$ are constant and the trivial solution, $W = \text{constant}$, is obtained. The above argument is also valid when finite negative powers of the parameter are considered in the trial solution, so the six types of factorization already include all possible kinds of SIPs with parameters related by translation.

4. Exactness of the SWKB formula

The conventional argument about the exactness of the SWKB approximation for SIPs with unbroken SUSY is based on the preservation of the degeneracy of the spectrum in the SUSY transformations [4]. However, we are going to show that, by considering the generic form of the superpotential obtained from the factorization method as discussed above, the exactness of the approximation is readily guaranteed.

As stated before, all the shape invariant potentials can be classified into one of the six factorization schemes. In particular, type B, C, D and F TSIPs are just the respective limiting cases of type A or type E potentials. Therefore, it suffices to establish the exactness of the SWKB formula for these two types of TSIPs. Firstly, we consider the type A potentials. The SWKB integral for the n th state of the type A Hamiltonian is given by

$$\begin{aligned} I_n &= \int_{x_1}^{x_2} \sqrt{E_n(a_1) - W^2(x, a_1)} dx \\ &= \int_{x_1}^{x_2} \sqrt{E_n^{(1)} - \left[b\sqrt{k_1^2 - \tilde{\alpha}^2} + ak_1 \right]^2} dx \end{aligned} \quad (4.1)$$

where x_1 and x_2 are the turning points as defined in section 1. We intend to show that the SWKB quantization formula $I_n = n\pi$ indeed leads to an exact solution of energy, namely, $E_n^{(1)} = -\tilde{\alpha}^2[(n+a)^2 - a^2]$. The SWKB integral (4.1) can be evaluated by introducing a new variable $w = \sqrt{k_1^2(x) - \tilde{\alpha}^2} + k_1(x)$:

$$I_n = \int_{w_1}^{w_2} \frac{2 dw}{w(w^2 - \tilde{\alpha}^2)} \sqrt{E_n^{(1)} w^2 - \left[\frac{a+b}{2} w^2 + \frac{(a-b)\tilde{\alpha}^2}{2} \right]^2} \quad (4.2)$$

where w_1 and w_2 are, respectively, the images of x_1 and x_2 in the w -plane.

To evaluate the integral by means of residue calculus, we rewrite it as

$$\begin{aligned}
 I_n &= \frac{1}{2} \int_{C_1} f(w) \, dw \\
 &= \frac{1}{2} \int_{C_2} f(w) \, dw
 \end{aligned}
 \tag{4.3}$$

where

$$f(w) = \frac{i|a+b|}{w(w^2 - \tilde{\alpha}^2)} \sqrt{(w-w_1)(w-w_2)(w-w_3)(w-w_4)} \, dw
 \tag{4.4}$$

with

$$w_1 = \frac{-\sqrt{E_n^{(1)}} - \sqrt{E_n^{(1)} - (a^2 - b^2)\tilde{\alpha}^2}}{a+b}
 \tag{4.5}$$

$$w_2 = \frac{-\sqrt{E_n^{(1)}} + \sqrt{E_n^{(1)} - (a^2 - b^2)\tilde{\alpha}^2}}{a+b}
 \tag{4.6}$$

$$w_3 = \frac{\sqrt{E_n^{(1)}} - \sqrt{E_n^{(1)} - (a^2 - b^2)\tilde{\alpha}^2}}{a+b}
 \tag{4.7}$$

$$w_4 = \frac{\sqrt{E_n^{(1)}} + \sqrt{E_n^{(1)} - (a^2 - b^2)\tilde{\alpha}^2}}{a+b}
 \tag{4.8}$$

and, as shown in figure 1, C_1 (C_2) is the contour enclosing the cut from w_1 to w_2 (from w_3 to w_4). Here the proper choice of the Riemann sheet is such that I_n is non-negative. It is straightforward to show that

$$\begin{aligned}
 \int_{C_1+C_2} f(w) \, dw &= \int_{C-C_3-C_4-C_5} f(w) \, dw \\
 &= \int_C f(w) \, dw - 2\pi i \sum_{w_n=\pm\tilde{\alpha},0} \text{Res}[f(w), w_n]
 \end{aligned}
 \tag{4.9}$$

where C is a circle at infinity (see figure 1). From the standard method of residue calculus, we have

$$\int_C f(w) \, dw = -2\pi|a+b|
 \tag{4.10}$$

$$2\pi i \text{Res}[f(w), 0] = -2\pi \frac{|(a-b)\tilde{\alpha}^2|}{\tilde{\alpha}^2}
 \tag{4.11}$$

$$2\pi i \text{Res}[f(w), \pm\tilde{\alpha}] = \frac{2\pi i}{\tilde{\alpha}} \sqrt{E_n^{(1)} - a^2\tilde{\alpha}^2}.
 \tag{4.12}$$

Therefore, by virtue of the SWKB formula, $E_n^{(1)}$ satisfies

$$-\frac{4\pi i}{\tilde{\alpha}} \sqrt{E_n^{(1)} - a^2\tilde{\alpha}^2} = 2\pi \left[2n + |a+b| - \frac{|(a-b)\tilde{\alpha}^2|}{\tilde{\alpha}^2} \right].
 \tag{4.13}$$

The signs of the two terms $a+b$ and $(a-b)\tilde{\alpha}^2$ can be derived from the asymptotic values of the superpotential W at the two spatial boundaries (e.g. $x = \pm\infty$). From equation (1.11) and the normalizability of the ground state wavefunction $\psi_0^{(1)}(x)$, the two asymptotic values

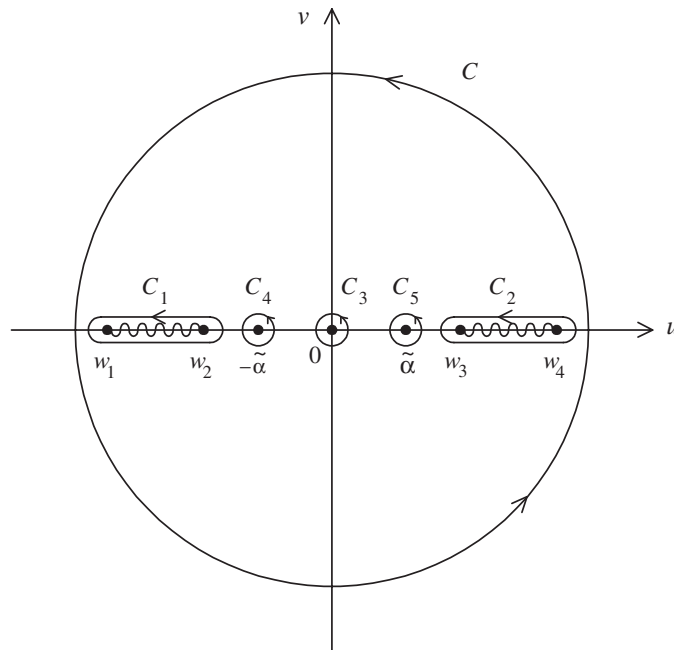


Figure 1. The contours in the complex $w = u + iv$ plane, which are used to evaluate the SWKB integral of type A potentials.

should be of opposite signs [2–5]. Together with equations (3.10), (3.13) for the case $s = 0$, and the fact that a is smaller (greater) than zero if $\tilde{\alpha}^2 > 0$ ($\tilde{\alpha}^2 < 0$), this readily shows that

$$a + b > 0 \quad (4.14)$$

$$(a - b)\tilde{\alpha}^2 < 0 \quad (4.15)$$

leading directly to the exact value of $E_n^{(1)}$:

$$E_n^{(1)} = -\tilde{\alpha}^2[(n + a)^2 - a^2]. \quad (4.16)$$

We will revert to this point in the next section to discuss the SWKB formula for BSUSY.

Meanwhile, we consider the SWKB integral for the type E problem, which can also be expressed in terms of a contour integral in another complex variable $z = k_1(x)$ (see figure 2):

$$I_n = \frac{1}{2} \int_{C_1} \frac{dz}{z^2 - \tilde{\alpha}^2} \sqrt{E_n^{(1)} - \left[\frac{b}{a} + az \right]^2} \quad (4.17)$$

with $a = a_1$. Evaluation of I_n is similar to the previous one. First, we deform the contour C_1 encircling the cut between z_1 to z_2 until it becomes a huge circle C with radius $R \rightarrow \infty$. Therefore, the contour integral for C_1 is equal to

$$\int_{C_1} g(z) dz = \int_C g(z) dz - 2\pi i \sum_{z_n = \pm \tilde{\alpha}} \text{Res}[g(z), z_n] \quad (4.18)$$

with

$$g(z) \equiv \frac{1}{2} \frac{dz}{z^2 - \tilde{\alpha}^2} \sqrt{E_n^{(1)} - \left[\frac{b}{a} + az \right]^2}. \quad (4.19)$$

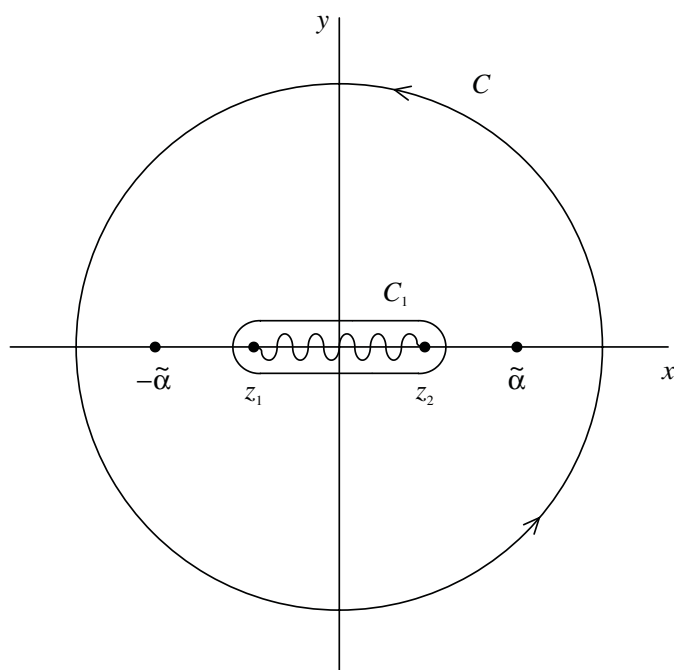


Figure 2. The contours in the complex $z = x + iy$ plane, which are used to evaluate the SWKB integral of type E potentials.

It can be shown that

$$-2\pi i \sum_{z_n = \pm \tilde{\alpha}} \text{Res}[g(z), z_n] = \frac{\pi i}{\tilde{\alpha}} \left[\sqrt{E_n^{(1)} - \left(\frac{b}{a} + a\tilde{\alpha}\right)^2} - \sqrt{E_n^{(1)} - \left(\frac{b}{a} - a\tilde{\alpha}\right)^2} \right] \quad (4.20)$$

and

$$\int_C g(z) dz = -2\pi a. \quad (4.21)$$

Therefore, by the SWKB formula, $E_n^{(1)}$ satisfies

$$\frac{\pi i}{\tilde{\alpha}} \left[\sqrt{E_n^{(1)} - \left(\frac{b}{a} + a\tilde{\alpha}\right)^2} - \sqrt{E_n^{(1)} - \left(\frac{b}{a} - a\tilde{\alpha}\right)^2} \right] = 2\pi(n + a) \quad (4.22)$$

leading directly to the exact value of $E_n^{(1)}$:

$$E_n(a_1) = \tilde{\alpha}^2 \left\{ a^2 - (a + n)^2 + b^2 \left[\frac{1}{a^2} - \frac{1}{(a + n)^2} \right] \right\}. \quad (4.23)$$

Since all TSIPs can be classified into the six different types mentioned above, which can all be represented by the type A and type E potentials, we can readily conclude from the results obtained above that the SWKB formula is exact for all bound states of TSIPs with unbroken SUSY.

5. SWKB formula for BSUSY

In this section we consider the exactness of the SWKB formula for BSUSY. In the BSUSY formalism, we have a freedom to choose the factorization energy ϵ that is defined through the relations

$$\tilde{A}^\dagger \tilde{A} \psi_n^{(1)} \equiv \left[-\hbar^2 \frac{d^2}{dx^2} + V_1(x) - \epsilon \right] \psi_n^{(1)} = [E_n^{(1)} - \epsilon] \psi_n^{(1)} \equiv \tilde{E}_n^{(1)} \psi_n^{(1)} \quad (5.1)$$

and

$$\tilde{A} \tilde{A}^\dagger \psi_n^{(2)} \equiv \left[-\hbar^2 \frac{d^2}{dx^2} + V_2(x) - \epsilon \right] \psi_n^{(2)} = [E_n^{(2)} - \epsilon] \psi_n^{(2)} \equiv \tilde{E}_n^{(2)} \psi_n^{(2)} \quad (5.2)$$

where

$$\tilde{A}^\dagger = -\hbar \frac{d}{dx} + \tilde{W}(x) \quad \tilde{A} = \hbar \frac{d}{dx} + \tilde{W}(x). \quad (5.3)$$

(Note that we have for the moment put back appropriate factors of \hbar in these equations for the purpose of comparison with those in section 1.) The new superpotential \tilde{W} therefore satisfies the equations

$$V_1(x) - \epsilon \equiv \tilde{V}_1(x) = \tilde{W}^2(x) - \hbar \tilde{W}'(x) \quad (5.4)$$

and

$$V_2(x) - \epsilon \equiv \tilde{V}_2(x) = \tilde{W}^2(x) + \hbar \tilde{W}'(x). \quad (5.5)$$

In addition, it is given by

$$\tilde{W}(x) = -\hbar \frac{[\psi_\epsilon(x)]'}{\psi_\epsilon(x)}. \quad (5.6)$$

Here $\psi_\epsilon(x)$ is the solution of the original Schrödinger equation for the factorization energy ϵ , namely

$$\left[-\hbar^2 \frac{d^2}{dx^2} + V_1(x) - \epsilon \right] \psi_\epsilon^{(1)} \equiv \tilde{A}^\dagger \tilde{A} \psi_\epsilon^{(1)} = 0. \quad (5.7)$$

In order to guarantee analyticity in $\tilde{W}(x)$ for BSUSY systems, the factorization energy ϵ is taken to be less than the original ground state energy $E_0^{(1)}$. Hence, $\psi_\epsilon(x)$ is not normalizable, resulting in the remarkable consequence that the asymptotic values of the superpotential \tilde{W} at the two spatial boundaries are of the same sign [4, 5].

Following from the standard arguments of SUSY QM [4, 5], in BSUSY systems the spectra of $\tilde{V}_1(x)$ and $\tilde{V}_2(x)$ are identical (exactly isospectral), namely $\tilde{E}_n^{(1)} = \tilde{E}_n^{(2)} > 0$ for $n = 0, 1, 2, \dots$. Owing to this remarkable fact, the eigenenergy of SIPs with BSUSY cannot be obtained directly from equation (1.15) and, instead, other methods were developed to evaluate the eigenenergies [17–20]. It then emerges as a surprise that the SWKB formula for BSUSY, equation (1.18), is exact for TSIPs [17–20]. In the following, we will show that the exactness of the SWKB formula for BSUSY is concomitant with that for unbroken SUSY.

First of all, we note that all known TSIPs with BSUSY belong to type A according to the classification scheme mentioned in section 3 [23, 17–20]. Therefore, it is legitimate to confine our discussion to type A TSIPs, whose superpotential \tilde{W} and \tilde{V}_1 , respectively, read

$$\tilde{W}(x, \tilde{a}_{s+1}) = \tilde{b} \sqrt{k_1^2 - \tilde{\alpha}^2} + \tilde{a} k_1 \quad (5.8)$$

$$\tilde{V}_1(x, \tilde{a}_{s+1}) = (\tilde{\alpha}^2 + \tilde{b}^2) k_1^2 - \tilde{b}^2 \tilde{\alpha}^2 + 2\tilde{a} \tilde{b} k_1 \sqrt{k_1^2 - \tilde{\alpha}^2} - \tilde{a} k_1' - \tilde{b} \frac{k_1 k_1'}{\sqrt{k_1^2 - \tilde{\alpha}^2}}. \quad (5.9)$$

Hereafter we adopt the abbreviation $\tilde{a} = \tilde{a}_{s+1}$ as long as there is no ambiguity. From equation (3.5), $k'_1 = k_1^2 - \tilde{\alpha}^2$, equation (5.9) can be simplified and acquires the form

$$\tilde{V}_1(x, \tilde{a}) = (\tilde{a}^2 - \tilde{a} + \tilde{b}^2)k_1^2 + (2\tilde{a}\tilde{b} - \tilde{b})k_1\sqrt{k_1^2 - \tilde{\alpha}^2} + (\tilde{a} - \tilde{b}^2)\tilde{\alpha}^2. \tag{5.10}$$

However, \tilde{V}_1 is related to the original potential V_1 , which preserves SUSY and is generated from the superpotential $W = b\sqrt{k_1^2 - \alpha^2} + ak_1$. Hence, by virtue of equation (5.4)

$$\begin{aligned} &(a^2 - a + b^2)k_1^2 + (2ab - b)k_1\sqrt{k_1^2 - \alpha^2} + (a - b^2)\alpha^2 \\ &= (\tilde{a}^2 - \tilde{a} + \tilde{b}^2)k_1^2 + (2\tilde{a}\tilde{b} - \tilde{b})k_1\sqrt{k_1^2 - \tilde{\alpha}^2} + (\tilde{a} - \tilde{b}^2)\tilde{\alpha}^2 + \epsilon. \end{aligned} \tag{5.11}$$

Comparing the coefficients, we obtain two relations between the two sets of parameters (a, b) and (\tilde{a}, \tilde{b}) :

$$a^2 - a + b^2 = \tilde{a}^2 - \tilde{a} + \tilde{b}^2 \tag{5.12}$$

$$2ab - b = 2\tilde{a}\tilde{b} - \tilde{b}. \tag{5.13}$$

These two equations have four sets of solutions, namely

$$\tilde{a} = a \qquad \tilde{b} = b \tag{5.14}$$

$$\tilde{a} = -a + 1 \qquad \tilde{b} = -b \tag{5.15}$$

$$\tilde{a} = b + \frac{1}{2} \qquad \tilde{b} = a - \frac{1}{2} \tag{5.16}$$

$$\tilde{a} = -b + \frac{1}{2} \qquad \tilde{b} = -a + \frac{1}{2}. \tag{5.17}$$

While the first set of solutions is trivial, the other three sets, respectively, represent three different potentials \tilde{V}_1 related to the original TSIP V_1 through equation (5.4), with the factorization energy being given by

$$\epsilon = \tilde{\alpha}^2(a^2 - \tilde{a}^2). \tag{5.18}$$

Hence, it is readily shown that

$$\tilde{E}_n^{(1)}(a) = -\tilde{\alpha}^2[(a+n)^2 - \tilde{a}^2]. \tag{5.19}$$

We note here that equation (5.19) is important in its own right. In addition to being the starting point for the ensuing discussion on the SWKB formula for BSUSY, it also serves as a means to evaluate the eigenenergy of TSIPs with either SUSY or BSUSY.

For concreteness, in the following discussion we assume (i) unbroken SUSY in the original potential $V_1(x, a_{s+1})$ and (ii) the existence of bound states in $V_1(x, a_s)$, $V_1(x, a_{s+1})$ and $V_1(x, a_{s+2})$ such that $a < -1/2$ ($a + b > 1$) and $b > 3/2$ ($a - b > 1$) for $\tilde{\alpha}^2 > 0$ ($\tilde{\alpha}^2 < 0$). By inspecting the behaviour of \tilde{W} at the spatial boundaries, it can be shown that the solutions in equations (5.14) and (5.15) preserve SUSY, whereas those represented by equations (5.16) and (5.17) result in BSUSY.

In order to evaluate the SWKB integral

$$\tilde{I}_n \equiv \int_{\tilde{x}_1}^{\tilde{x}_2} \sqrt{\tilde{E}_n^{(1)} - \tilde{W}^2(x)} dx \tag{5.20}$$

we apply the contour integration method detailed in section 4 to arrive at the following result:

$$-\frac{4\pi i}{\tilde{\alpha}} \sqrt{\tilde{E}_n^{(1)} - \tilde{a}^2\tilde{\alpha}^2} = 4\tilde{I}_n + 2\pi \left[|\tilde{a} + \tilde{b}| - \frac{|(\tilde{a} - \tilde{b})\tilde{\alpha}^2|}{\tilde{\alpha}^2} \right]. \tag{5.21}$$

Since a is smaller (greater) than zero if $\tilde{\alpha}^2 > 0$ ($\tilde{\alpha}^2 < 0$), for BSUSY represented by equations (5.16) proper conditions on the asymptotic behaviour of \tilde{W} at the spatial boundaries lead to

$$\tilde{a} + \tilde{b} = a + b > 0 \quad (5.22)$$

$$(\tilde{a} - \tilde{b})\tilde{\alpha}^2 = (b - a + 1)\tilde{\alpha}^2 < 0 \quad (5.23)$$

and

$$|\tilde{a} + \tilde{b}| - \frac{|(\tilde{a} - \tilde{b})\tilde{\alpha}^2|}{\tilde{\alpha}^2} = 2\tilde{b} \quad (5.24)$$

thereby showing that

$$\begin{aligned} \tilde{E}_n^{(1)} &= -\tilde{\alpha}^2 \left[\left(\frac{\tilde{I}_n}{\pi} + \tilde{b} \right)^2 - \tilde{a}^2 \right] \\ &= -\tilde{\alpha}^2 \left[\left(\frac{\tilde{I}_n}{\pi} + a - \frac{1}{2} \right)^2 - \tilde{a}^2 \right]. \end{aligned} \quad (5.25)$$

As a consequence, exactness of the SWKB formula is then established if

$$\tilde{I}_n \equiv \int_{x_1}^{x_2} \sqrt{E_n^{(1)} - W^2(x)} dx = \left(n + \frac{1}{2} \right) \pi \quad \text{for } n = 0, 1, 2, \dots \quad (5.26)$$

Likewise, for the situation associated with equation (5.17) the following result can be obtained:

$$\begin{aligned} \tilde{E}_n^{(1)} &= -\tilde{\alpha}^2 \left[\left(\frac{\tilde{I}_n}{\pi} - \tilde{b} \right)^2 - \tilde{a}^2 \right] \\ &= -\tilde{\alpha}^2 \left[\left(\frac{\tilde{I}_n}{\pi} + a - \frac{1}{2} \right)^2 - \tilde{a}^2 \right] \end{aligned} \quad (5.27)$$

which shows that equation (5.26) and hence the BSUSY SWKB formula are generally correct.

6. Multi-parameter SIPs

In the above discussion, we have deliberately limited our discussion to TSIPs with one parameter. However, we will now show that TSIPs with multiple parameters are in fact representable in terms of the one-parameter TSIPs. Therefore, our method for showing the exactness of the SWKB formula also applies to TSIPs with multiple parameters. In the following, we first introduce the concept of multi-parameter TSIPs and demonstrate the equivalence of multi-parameter TSIPs and one-parameter TSIPs.

A TSIP with multiple parameters is characterized by a set of n parameters [24, 25], i.e.

$$V = V(a) \quad \text{where } a = \{a_1, a_2, a_3, \dots, a_n\}. \quad (6.1)$$

The shape invariance condition for the multi-parameter case is very similar to the one-parameter case but the function $f(a)$ now maps the whole set of parameters from a to \bar{a} :

$$f(a) = \bar{a} = \{\bar{a}_1, \bar{a}_2, \bar{a}_3, \dots, \bar{a}_n\}. \quad (6.2)$$

For a translational SIP, the two sets of parameters are related by

$$\bar{a}_i = a_i + \epsilon_i \quad \text{for all } i = 1, 2, \dots, n \quad (6.3)$$

where ϵ_i are some nonzero constants. We can renormalize each parameter in units of ϵ_i and hence introduce the new parameters

$$\gamma = \{m_1, m_2, m_3, \dots, m_n\} \quad \text{where } m_i = a_i/\epsilon_i \quad (6.4)$$

and

$$\bar{\gamma} = f(\gamma) = \{\bar{m}_1, \bar{m}_2, \bar{m}_3, \dots, \bar{m}_n\} \quad \text{where} \quad \bar{m}_i = m_i + 1. \quad (6.5)$$

Furthermore, we define

$$\tilde{m}_i \equiv m_i - M \quad \text{for all} \quad i = 1, 2, \dots, n \quad (6.6)$$

where $M = \sum_{i=1}^n c_i m_i$ with $\sum_{i=1}^n c_i = 1$. Obviously, \tilde{m}_i is invariant under the transformation $m_i \rightarrow m_i + 1$:

$$\tilde{m}_i \rightarrow \tilde{m}'_i = m_i + 1 - \sum_{i=1}^n c_i (m_i + 1) = m_i - M = \tilde{m}_i. \quad (6.7)$$

In terms of \tilde{m}_i and M , the transformation $m_i \rightarrow m_i + 1$ is equivalent to the transformation $M \rightarrow M + 1$:

$$m_i + 1 = \tilde{m}_i + M + 1. \quad (6.8)$$

Therefore, all multi-parameter TSIPs can actually reduce to one-parameter TSIPs.

In fact, Cariñena and Ramos [25] have demonstrated that all multi-parameter superpotentials of the form

$$W(x, \gamma) = k_0(x) + \sum_{i=1}^n m_i k_i(x) \quad (6.9)$$

would eventually reduce to the one-parameter form (3.3). They have also tried many other generalizations of SIPs which depend on n parameters transformed by means of translation and found no nontrivial solutions for $n > 1$.

7. Discussion

The exactness of the SWKB formula for TSIPs is a rather surprising result because the SWKB formula is derived from the first-order WKB formula that is known to be inexact even for most solvable potentials. Despite the fact that it was first discovered almost two decades ago [9], the validity of its original proof, which predicted the exactness of the SWKB formula for all SIPs, is obviously questionable. In this paper, we point out a logical inconsistency in the original proof [9], which has not yet been noted in the literature. In addition, we provide a direct proof to show that the translational condition on the superpotential of TSIPs readily guarantees the exactness of the SWKB formula. It is remarkable that both the analytic expression for the eigenenergy and the exactness of the SWKB formula originate directly from the canonical form of the superpotential of TSIPs. Hence, the exactness of the SWKB formula is by no means a generic behaviour for all SIPs. In fact, it fails to yield exact values of the eigenenergy for SIPs related by scaling [12], demonstrating that the exactness of the SWKB formula crucially relies on the detailed form of the superpotential, instead of shape invariance itself.

Our method can readily be generalized to discuss the exactness of the SWKB formula for the BSUSY case [17–20]. To the best of our knowledge, a unified robust proof for the exactness of the SWKB formula for both the unbroken and broken SUSY cases has not yet been formulated. The approach developed in this paper then bridges the gap between the two different situations. In particular, it points out that the two different forms of the SWKB formula, respectively applying to SUSY and BSUSY, are closely related to the asymptotic behaviour of the superpotentials W and \tilde{W} . Lastly, we establish the equivalence of one-parameter and multi-parameter TSIPs, thus in turn showing the exactness of the SWKB formula for multi-parameter TSIPs.

Acknowledgments

We thank an anonymous referee for drawing our attention to the SWKB formula for the BSUSY case, which has significantly broadened the scope of our paper. Our work is supported in part by a grant from the Hong Kong Research Grants Council (grant no CUHK4282/00P).

References

- [1] Witten E 1981 *Nucl. Phys. B* **188** 513
- [2] Cooper F and Freedman B 1983 *Ann. Phys., NY* **146** 262
- [3] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917
- [4] Cooper F, Khare A and Sukhatme U P 1995 *Phys. Rep.* **251** 267
- [5] Junker G 1996 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer)
- [6] Gendenshtein F 1983 *JETP Lett.* **38** 356
- [7] Barclay D T, Dutt R, Gangopadhyaya A, Khare A, Pagnamenta A and Sukhatme U 1993 *Phys. Rev. A* **48** 2786
- [8] Khare A and Sukhatme U 1993 *J. Phys. A: Math. Gen.* **26** L901
- [9] Comtet A, Bandrauk A D and Campbell D K 1985 *Phys. Lett. B* **150** 159
- [10] Fröman N and Fröman P O 1965 *JWKB Approximation, Contributions to the Theory* (Amsterdam: North-Holland)
- [11] Dutt R, Khare A and Sukhatme U 1986 *Phys. Lett. B* **181** 295
- [12] Barclay D T, Khare A and Sukhatme U 1993 *Phys. Lett. A* **183** 263
- [13] Khare A 1985 *Phys. Lett. B* **161** 131
- [14] Khare A and Varshni Y P 1989 *Phys. Lett. A* **142** 1
- [15] Barclay D T and Maxwell C J 1991 *Phys. Lett. A* **157** 357
- [16] Milczarski P and Giller 2000 *J. Phys. A: Math. Gen.* **33** 357
- [17] Inomata A and Junker G 1992 Quasi-classical approach to path integrals in supersymmetric quantum mechanics *Lectures on Path Integrals (Trieste 1991)* ed H Cerdeira *et al* (Singapore: World Scientific)
- [18] Inomata A, Junker G and Suparmi A 1993 *J. Phys. A: Math. Gen.* **26** 2261
- [19] Dutt R, Gangopadhyaya A, Khare A, Pagnamenta A and Sukhatme U 1993 *Phys. Lett. A* **174** 363
- [20] Dutt R, Gangopadhyaya A, Khare A, Pagnamenta A and Sukhatme U 1993 *Phys. Rev. A* **48** 1845
- [21] Schrödinger E 1940 *Proc. R. Ir. Acad. A* **46** 9
- [22] Schrödinger E 1941 *Proc. R. Ir. Acad. A* **46** 183
- [23] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
- [24] Cooper F, Ginocchio J N and Khare A 1987 *Phys. Rev. D* **36** 2458
- [25] Cariñena J F and Ramos A 2000 *J. Phys. A: Math. Gen.* **33** 3467
- [26] Gangopadhyaya A, Panigrahi P and Sukhatme U 1994 *Helv. Phys. Acta. A* **67** 363
- [27] Nieto M M 1979 *Am. J. Phys.* **47** 1067